Vacuum stability conditions of the general two-Higgs-doublet potential*

Yisheng Song†

Abstract

In this paper, we present the novel analytical expressions for the bounded-from-below or the vacuum stability conditions of scalar potential for a general CP violating two-Higgs-doublet model by using the concepts of co-positivity and the gauge orbit spaces. More precisely, several analyticial sufficient conditions and necessary conditions are established for the vacuum stability of the general 2HDM potential, respectively. We also give an equivalent condition of the vacuum stability of the general 2HDM potential in theory, and then, apply it to derive the analyticial necessary conditions of the general 2HDM potential. Meanwhile, the semi-positive definiteness is proved for a class of 4th-order 2-dimensional complex tensor.

Keywords: Co-positivity, Complex tensors, CP violation, 2HDM

1 Introduction

In 1973, the fist two-Higgs-doublet model (for short, 2HDM) is presented by Lee [1, 2]. Subsequently, Weinberg [3] proposed a general multi-Higgs potential model. Since then, the stability of the scalar Higgs potential is an important problem with the Standard Model (for short, SM) at high-energies. The bounded-from-below (for short, BFB) or the vacuum stability of SM is very noticeable in particle physics community. One of the simplest extensions of the SM Higgs sector is the 2HDM [1, 4]. It is well-known that the most general Higgs potential

*This work was supported by the National Natural Science Foundation of P.R. China (Grant No.12171064), by The team project of innovation leading talent in chongqing (No.CQYC20210309536) and by the Foundation of Chongqing Normal university (20XLB009)

†Corresponding author. School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331 P.R. China, Email: yisheng.song@cqnu.edu.cn

for a 2HDM with Higgs doublets Φ_1 and Φ_2 can be written [5–7]

$$\begin{aligned}V_H(\Phi_1,\Phi_2) =&\mu_{11}\Phi_1^*\Phi_1+\mu_{22}\Phi_2^*\Phi_2-(\mu_{12}\Phi_1^*\Phi_2+\mu_{12}^*\Phi_2^*\Phi_1)\\&+\lambda_1(\Phi_1^*\Phi_1)^2+\lambda_2(\Phi_2^*\Phi_2)^2\\&+\lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2)+\lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1)\\&+\frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2+\frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2\\&+(\Phi_1^*\Phi_1)(\lambda_6\Phi_1^*\Phi_2+\lambda_6^*\Phi_2^*\Phi_1)\\&+(\Phi_2^*\Phi_2)(\lambda_7\Phi_1^*\Phi_2+\lambda_7^*\Phi_2^*\Phi_1),\end{aligned}\tag{1}$$

where Φ^* is Hermitian conjugate of Φ. The parameters μ_{11}, μ_{22} and λ_i $(i=1,2,3,4)$ are real, μ_{12} and λ_i $(i=5,6,7)$ are complex. If such a 2HDM has explicit CP conservation, then μ_{12} and λ_i $(i=5,6,7)$ are real.

The Higgs potential of such a 2HDM (1) may be said [6,8]

$$V_H(\Phi_1,\Phi_2)=\sum_{a,b=1}^{2}\mu_{ab}\Phi_a^*\Phi_b+\sum_{i,j,k,l=1}^{2}t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l),\tag{2}$$

where, by definition,

$$t_{ijkl}=t_{klij},\ t_{ijkl}=t_{jilk}^*,\ \mu_{ab}=\mu_{ba}^*.\tag{3}$$

The quartic part,

$$V_4(\Phi_1,\Phi_2)=\sum_{i,j,k,l=1}^{2}t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l),\tag{4}$$

gives a 4th-order 2-dimensional complex tensor $\mathcal{T}=(t_{ijkl})$:

$$\begin{aligned}t_{1111}=&\lambda_1,\ t_{2222}=\lambda_2,\\t_{1122}=&t_{2211}=\frac{1}{2}\lambda_3,\ t_{1221}=t_{2112}=\frac{1}{2}\lambda_4,\\t_{1212}=&\frac{1}{2}\lambda_5,\ t_{2121}=\frac{1}{2}\lambda_5^*,\\t_{1112}=&t_{1211}=\frac{1}{2}\lambda_6,\ t_{1121}=\ t_{2111}=\frac{1}{2}\lambda_6^*,\\t_{1222}=&t_{2212}=\frac{1}{2}\lambda_7,\ t_{2122}=\ t_{2221}=\frac{1}{2}\lambda_7^*.\end{aligned}\tag{5}$$

The stability of the 2HDM potential requires that there is no direction in field space along which the potential tends to minus infinity, i.e., it is the BFB. In general, the quartic part of the scalar potential, V_4, is non-negative for arbitrarily large values of the component fields, but the quadratic part of the scalar potential, V_2, can take negative

values for at least some values of the fields [6]. Considering only the quartic part V_4, the condition for stability (the BFB) of the scalar potential in the 2HDM is equivalent to the co-positivity or semi-positive definiteness of the tensor $\mathcal{T} = (t_{ijkl})$ given by the Higgs quartic coupling λ_i, i.e. $V_4(\Phi_1, \Phi_2) \geq 0$. When $\lambda_6 = \lambda_7 = 0$ and λ_5 is real, the vacuum stability conditions of 2HDM potential [4, 9, 10, 12–14] are the following:

$$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0, \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

In 2016, Kannike [15–17] presented the vacuum stability conditions of the scalar potential of two Higgs doublets in the 2HDM with explicit CP conservation. Chauhan [18] derived analytic necessary and sufficient conditions for the vacuum stability of the left-right symmetric model, and gave the sufficient conditions for successful symmetry breaking. Recently, Song [19] showed the analytical sufficient and necessary conditions of the co-positivity of the tensor $\mathcal{T} = (t_{ijkl})$ with the real nembers λ_i ($i = 5, 6, 7$), and moreover, the vacuum stability conditions of scalar potential for the 2HMD with explicit CP conservation was obtained. For more details about the BFB or the vacuum stability conditions of the 2HDM potential, see Refs. [10, 11, 20, 21] for 2HDM with CP conservation; Refs. [7, 21] for the most general 2HDM; Refs. [6, 21, 22] for 2HDM with CP conservation and CP violation; Ref. [23] for 2HDM handled numerically and others references that are no cited here.

In this paper, we provide three new analytical sufficient conditions for the bounded-from-below or the vacuum stability of scalar potential for a general 2HDM by using the co-positivity of 4th-order 2-dimensional symmetric real tensor. A sufficient and necessary condition of the vacuum stability of the general 2HDM potential is given in theory, which contains the vacuum stability condition of the general 2HDM potential with $\mathbb{Z}_2$ symmetry as a special case. Then, we apply this conclusion to derive the analyticial analyticial necessary conditions of the vacuum stability of a general 2HDM scalar potential. Meanwhile, the analytical sufficient conditions and necessary conditions are obtained for the semi-positive definiteness of a class of 4th-order 2-dimensional complex tensor.

2 Co-positivity criteria

The co-positivity of a matrix $\mathbf{M} = (\mu_{ij})$ has been applied to test the vacuum stability of the 2HDM in Refs. [15–18]. It is kown-well that a 2×2 symmetric real matrix $\mathbf{M} = (\mu_{ij})$ is co-positive, i.e., for all non-negative vectors $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, the quadratic form

$$\mathbf{x}^\top \mathbf{M}\mathbf{x} = \mu_{11}x_1^2 + 2\mu_{12}x_1x_2 + \mu_{22}x_2^2 \geq 0,$$

if and only if [24–26]

$$\mu_{11} \geq 0, \mu_{22} \geq 0 \text{ and } \mu_{12} + \sqrt{\mu_{11}\mu_{22}} \geq 0. \tag{6}$$

The co-positivity of a symmetric real tensor has been used to the SM in literature to obtain vacuum stability conditions in Refs. [15, 19, 27–30]. A 4th-order n-dimensional symmetric real tensor $\mathcal{T} = (t_{ijkl})$ is co-positive [31–36] if for all non-negative vectors $\mathbf{x} = (x_1, x_2, \cdots x_n)^\top \in \mathbb{R}^n$, the quartic form

$$\mathcal{T}\mathbf{x}^4 = \sum_{i,j,k,l=1}^{n} t_{ijkl} x_i x_j x_k x_l \geq 0.$$

Let $t_{1111} = \alpha_0 > 0$ and $t_{2222} = \alpha_4 > 0$. Recently, Song and Li [28] presented the analytical expressions of co-positivity of a 4th-order 2-dimensional symmetric $\mathcal{T}$ with the help of the update version ([33]) of Ulrich and Watson's result [37]. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric real tensor with its entires

$$t_{1111} = \alpha_0, t_{2222} = \alpha_4, t_{1112} = \frac{1}{4}\alpha_1, t_{1122} = \frac{1}{6}\alpha_2, t_{1222} = \frac{1}{4}\alpha_3.$$

Then the quartic form

$$\mathcal{T}\mathbf{x}^4 = \alpha_0 x_1^4 + \alpha_1 x_1^3 x_2 + \alpha_2 x_1^2 x_2^2 + \alpha_3 x_1 x_2^3 + \alpha_4 x_2^4 \geq 0 \tag{7}$$

for all $x_1 \geq 0, x_2 \geq 0$ if and only if

$$\begin{aligned}
&(1)\Delta \leq 0, \alpha_1\sqrt{\alpha_4} + \alpha_3\sqrt{\alpha_0} > 0;\\
&(2)\alpha_1 \geq 0, \alpha_3 \geq 0, 2\sqrt{\alpha_0\alpha_4} + \alpha_2 \geq 0;\\
&(3)\Delta \geq 0, |\alpha_1\sqrt{\alpha_4} - \alpha_3\sqrt{\alpha_0}| \leq 4\sqrt{\alpha_0\alpha_2\alpha_4 + 2\alpha_0\alpha_4\sqrt{\alpha_0\alpha_4}},\\
&\qquad (i) - 2\sqrt{\alpha_0\alpha_4} \leq \alpha_2 \leq 6\sqrt{\alpha_0\alpha_4},\\
&\qquad (ii)\alpha_2 > 6\sqrt{\alpha_0\alpha_4}\\
&\qquad \alpha_1\sqrt{\alpha_4} + \alpha_3\sqrt{\alpha_0} \geq -4\sqrt{\alpha_0\alpha_2\alpha_4 - 2\alpha_0\alpha_4\sqrt{\alpha_0\alpha_4}},
\end{aligned}$$

where $\Delta = 4(12\alpha_0\alpha_4 - 3\alpha_1\alpha_3 + \alpha_2^2)^3 - (72\alpha_0\alpha_2\alpha_4 + 9\alpha_1\alpha_2\alpha_3 - 2\alpha_2^3 - 27\alpha_0\alpha_3^2 - 27\alpha_1^2\alpha_4)^2$.

Song and Qi [29, Theorem 3.7] gave a stronger sufficient condition for the co-positivity of a symmetric real tensor $\mathcal{T} = (t_{ijkl})$. That is, $\mathcal{T}\mathbf{x}^4 \geq 0$ for all $x_1 \geq 0, x_2 \geq 0$ if

$$\begin{aligned}
&\beta = \alpha_1 + 4\sqrt[4]{\alpha_0^3\alpha_4} \geq 0, \gamma = \alpha_3 + 4\sqrt[4]{\alpha_0\alpha_4^3} \geq 0,\\
&\alpha_2 - 6\sqrt{\alpha_0\alpha_4} + 2\sqrt{\beta\gamma} \geq 0.
\end{aligned} \tag{8}$$

Song [19] obtained an analytical sufficient and necessary condition for the co-positivity of a symmetric real tensor $\mathcal{T}(\rho,\theta) = (t_{ijkl}(\rho,\theta))$ with two parameters $\rho \in [0,1]$ and $\theta \in \mathbb{R}$,

$$\begin{aligned} t_{1111} =& \Lambda_1, t_{2222} = \Lambda_2, \\ t_{1122} =& \frac{1}{6}(\Lambda_3 + \Lambda_4\rho^2 + \Lambda_5\rho^2\cos 2\theta), \\ t_{1112} =& \frac{1}{2}\Lambda_6\rho\cos\theta,\ t_{1222} = \frac{1}{2}\Lambda_7\rho\cos\theta. \end{aligned} \tag{9}$$

That is, $\Lambda_1 > 0$, $\Lambda_2 > 0$, the quartic form

$$\begin{aligned} \mathcal{T}(\rho,\theta)\mathbf{x}^4 =& \Lambda_1 x_1^4 + \Lambda_2 x_2^4 + (\Lambda_3 + \Lambda_4\rho^2 + \Lambda_5\rho^2\cos 2\theta)x_1^2x_2^2 \\ &+ 2(\rho\Lambda_6\cos\theta)x_1^3x_2 + 2(\rho\Lambda_7\cos\theta)x_1x_2^3 \geq 0, \end{aligned} \tag{10}$$

for all $x_1 \geq 0, x_2 \geq 0$ if and only if

(a) $\Lambda_6 = \Lambda_7 = 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 - |\Lambda_5| + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$;

(b) $\Delta \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$,

$|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}}$,

(i) $-2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2}$,

(ii) $\Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2}$ and

$|\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}}$,

where $\Delta = 4(12\Lambda_1\Lambda_2 - 12\Lambda_6\Lambda_7 + (\Lambda_3+\Lambda_4+\Lambda_5)^2)^3 - (72\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) + 36\Lambda_6\Lambda_7(\Lambda_3+\Lambda_4+\Lambda_5) - 2(\Lambda_3+\Lambda_4+\Lambda_5)^3 - 108\Lambda_1\Lambda_7^2 - 108\Lambda_6^2\Lambda_2)^2$.

3 Vacuum stability of the general 2HDM potential

3.1 Sufficient conditions

In this section, we mainly give the vacuum stability conditions of the 2HDM potential (1) with explicit CP violation. We rewrite the quartic

part of such a 2HDM potential as follow

$$
\begin{aligned}
V_4(\Phi_1,\Phi_2)=&\sum_{i,j,k,l=1}^{2} t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l)\\
=&\lambda_1(\Phi_1^*\Phi_1)^2+\lambda_2(\Phi_2^*\Phi_2)^2\\
&+\lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2)+\lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1)\\
&+\frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2+\frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2\\
&+(\Phi_1^*\Phi_1)(\lambda_6\Phi_1^*\Phi_2+\lambda_6^*\Phi_2^*\Phi_1)\\
&+(\Phi_2^*\Phi_2)(\lambda_7\Phi_1^*\Phi_2+\lambda_7^*\Phi_2^*\Phi_1).
\end{aligned}
\tag{11}
$$

Let ϕ_i=$|\Phi_i|=\sqrt{\Phi_i^*\Phi_i}$, the modulus of Φ_i for $i=1,2$. Then

$$\Phi_1^*\Phi_2=\phi_1\phi_2\rho e^{i\theta} \text{ and } \Phi_2^*\Phi_1=\phi_1\phi_2\rho e^{-i\theta},$$

here $i^2=-1$ and $\rho\in[0,1]$ is the orbit space parameter [9, 15, 18]. Let

$$\lambda_5=|\lambda_5|e^{i\varphi_5}, \lambda_6=|\lambda_6|e^{i\varphi_6}, \lambda_7=|\lambda_7|e^{i\varphi_7},$$

where φ_k is argument of the complex number λ_k $(k=5,6,7)$. Then

$$\lambda_5^*=|\lambda_5|e^{-i\varphi_5}, \lambda_6^*=|\lambda_6|e^{-i\varphi_6}, \lambda_7^*=|\lambda_7|e^{-i\varphi_7}.$$

So, we have

$$
\begin{aligned}
V_4(\Phi_1,\Phi_2)=&\lambda_1\phi_1^4+\lambda_2\phi_2^4+\lambda_3\phi_1^2\phi_2^2+\lambda_4\rho^2\phi_1^2\phi_2^2\\
&+\frac{|\lambda_5|}{2}(e^{i(\varphi_5+2\theta)}+e^{-i(\varphi_5+2\theta)})\phi_1^2\phi_2^2\rho^2\\
&+|\lambda_6|(e^{i(\varphi_6+\theta)}+e^{-i(\varphi_6+\theta)})\phi_1^3\phi_2\rho\\
&+|\lambda_7|(e^{i(\varphi_7+\theta)}+e^{-i(\varphi_7+\theta)})\phi_1\phi_2^3\rho\\
=&\lambda_1\phi_1^4+\lambda_2\phi_2^4+\lambda_3\phi_1^2\phi_2^2+\lambda_4\rho^2\phi_1^2\phi_2^2\\
&+|\lambda_5|\phi_1^2\phi_2^2\rho^2\cos(\varphi_5+2\theta)\\
&+2|\lambda_6|\phi_1^3\phi_2\rho\cos(\varphi_6+\theta)\\
&+2|\lambda_7|\phi_1\phi_2^3\rho\cos(\varphi_7+\theta)\\
=&\lambda_1\phi_1^4+\lambda_2\phi_2^4+\lambda_3\phi_1^2\phi_2^2+\lambda_4\rho^2\phi_1^2\phi_2^2\\
&+|\lambda_5|\phi_1^2\phi_2^2\rho^2(\cos\varphi_5\cos 2\theta-\sin\varphi_5\sin 2\theta)\\
&+2|\lambda_6|\phi_1^3\phi_2\rho(\cos\varphi_6\cos\theta-\sin\varphi_6\sin\theta)\\
&+2|\lambda_7|\phi_1\phi_2^3\rho(\cos\varphi_7\cos\theta-\sin\varphi_7\sin\theta).
\end{aligned}
$$

Obviously, $\mathbf{Re}\lambda_k=|\lambda_k|\cos\varphi_k, \mathbf{Im}\lambda_k=|\lambda_k|\sin\varphi_k, (k=5,6,7)$. Then,

noticing $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$\begin{aligned}V_4(\Phi_1,\Phi_2) =&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4\rho^2 + \mathbf{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2\\&+ 2(\rho\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\rho\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1\\&- 2\rho\phi_1\phi_2(\mathbf{Im}\lambda_5\rho\phi_1\phi_2\sin\theta\cos\theta\\&\qquad + \mathbf{Im}\lambda_6\phi_1^2\sin\theta + \mathbf{Im}\lambda_7\phi_2^2\sin\theta)\\=&V_4'(\phi_1,\phi_2) + V_4''(\phi_1,\phi_2),\end{aligned} \tag{12}$$

where

$$\begin{aligned}V_4'(\phi_1,\phi_2) =&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4\rho^2 + \mathbf{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2\\&+ 2(\rho\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\rho\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1,\\V_4''(\phi_1,\phi_2) =& - 2(\rho\sin\theta)[(\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2\\&\qquad + \mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2]\phi_1\phi_2.\end{aligned} \tag{13}$$

Applying the co-positivity of a real tensor (9) with

$$\Lambda_i = \lambda_i \ (i = 1,2,3,4), \Lambda_k = \mathbf{Re}\lambda_k \ (k = 5,6,7)$$

to obtain that $\lambda_1 > 0, \lambda_2 > 0$,

$$V_4'(\phi_1,\phi_2) \geq 0 \text{ for all } \phi_1,\phi_2 \tag{14}$$

if and only if

(I) $\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0,$

$\lambda_3 + \lambda_4 - |\mathbf{Re}\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0;$

(II) $\Delta \geq 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0,$

$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},$

(i) $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},$

(ii) $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$ and

$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},$

where $\Delta = 4(12\lambda_1\lambda_2 - 12\mathbf{Re}\lambda_6\mathbf{Re}\lambda_7 + (\lambda_3+\lambda_4+\mathbf{Re}\lambda_5)^2)^3 - (72\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 36\mathbf{Re}\lambda_6\mathbf{Re}\lambda_7(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5)^3 - 108\lambda_1(\mathbf{Re}\lambda_7)^2 - 108(\mathbf{Re}\lambda_6)^2\lambda_2)^2$.

After making simple calculations ($\sin\theta \neq 0$), we have

$$V_4''(\phi_1,\phi_2) \geq 0 \text{ for all } \phi_1,\phi_2 \tag{15}$$

if and only if

$$\begin{cases}\mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \geq 0, \ \sin\theta < 0,\\ \mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \leq 0, \ \sin\theta > 0.\end{cases}$$

By the co-positivity of a real matrix (6) with

$$\mu_{11} = \mathbf{Im}\lambda_6, \mu_{12} = (\rho\cos\theta)\mathbf{Im}\lambda_5, \mu_{22} = \mathbf{Im}\lambda_7,$$

we obtain that

$$\mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \geq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$\mathbf{Im}\lambda_6 \geq 0, \mathbf{Im}\lambda_7 \geq 0, (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Case 1: $\mathbf{Im}\lambda_5 \geq 0$.

The function $g(\rho\cos\theta) = (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$ reaches its minimum at $\rho\cos\theta = -1$, so for all $\rho\cos\theta \in [-1, 1]$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Case 2: $\mathbf{Im}\lambda_5 < 0$.

The function $g(\rho\cos\theta) = (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$ reaches its minimum at $\rho\cos\theta = 1$, so for all $\rho\cos\theta \in [-1, 1]$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow \mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Therefore, for any real number $\mathbf{Im}\lambda_5$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Similarly, we also have

$$\mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \leq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$-\mathbf{Im}\lambda_6\phi_1^2 - (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 - \mathbf{Im}\lambda_7\phi_2^2 \geq 0 \text{ for all } \phi_1, \phi_2$$

which is equivalent to the co-positivity of 2×2 matrix $M = (\mu_{ij})$ with its entries,

$$\mu_{11} = -\mathbf{Im}\lambda_6, \mu_{12} = -(\rho\cos\theta)\mathbf{Im}\lambda_5, \mu_{22} = -\mathbf{Im}\lambda_7.$$

This means that

$$-\mathbf{Im}\lambda_6 \geq 0, -\mathbf{Im}\lambda_7 \geq 0, -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0,$$

that is,

$$\mathbf{Im}\lambda_6 \leq 0, \mathbf{Im}\lambda_7 \leq 0, -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Similarly, the function

$$f(\rho\cos\theta) = -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$$

reaches its minimum at $\rho\cos\theta = 1$ ($\mathbf{Im}\lambda_5 \geq 0$) or -1 ($\mathbf{Im}\lambda_5 < 0$), and hence, for any real number $\mathbf{Im}\lambda_5$,

$$-(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

So, we get the conclusion that

$$V_4''(\phi_1, \phi_2) \geq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$\textbf{(III)} \qquad \mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7 \geq 0,\ -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

In summary, we prove the analytic conditions (**I**), (**II**) and (**III**) assure the vacuum stability of the 2HDM potential with explicit CP violation. At the same time, we also obtain the semi-positive definiteness of a 4th-order 2-dimensional complex tensor $\mathcal{T} = (t_{ijkl})$ definited by the Eq. (5).

In term of Eq. (8), we also may obtain a stronger sufficient condition. That is, $V_4(\phi_1, \phi_2) \geq 0$ for all ϕ_1, ϕ_2 if for all $\rho \in [0, 1]$ and all $\theta \in [0, 2\pi]$,

$$\begin{aligned}
&\beta(\theta) = 2(\rho|\lambda_6|\cos(\varphi_6 + \theta)) + 4\sqrt[4]{\lambda_1^3\lambda_2} \geq 0,\\
&\gamma(\theta) = 2(\rho|\lambda_7|\cos(\varphi_7 + \theta)) + 4\sqrt[4]{\lambda_1\lambda_2^3} \geq 0,\\
&(\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\varphi_5 + 2\theta)) - 6\sqrt{\lambda_1\lambda_2} + 2\sqrt{\beta(\theta)\gamma(\theta)} \geq 0.
\end{aligned}$$

Which is equivalent to

$$\textbf{(IV)} \qquad \begin{aligned}
&\beta = 2\sqrt[4]{\lambda_1^3\lambda_2} - |\lambda_6| \geq 0, \gamma = 2\sqrt[4]{\lambda_1\lambda_2^3} - |\lambda_7| \geq 0,\\
&\lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0,\\
&\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0.
\end{aligned}$$

Similarly, $V_4'(\phi_1, \phi_2) \geq 0$ for all ϕ_1, ϕ_2 if

$$\textbf{(IV}'\textbf{)} \qquad \begin{aligned}
&\beta' = 2\sqrt[4]{\lambda_1^3\lambda_2} - |\mathbf{Re}\lambda_6| \geq 0, \gamma' = 2\sqrt[4]{\lambda_1\lambda_2^3} - |\mathbf{Re}\lambda_7| \geq 0,\\
&\lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta'\gamma'} \geq 0,\\
&\lambda_3 + \lambda_4 - |\mathbf{Re}\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta'\gamma'} \geq 0.
\end{aligned}$$

Remark 3.1. Four analyticial sufficient conditions are following:

(1) $\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0$, the conditions (**I**) and (**III**);

(2) $\mathbf{Re}\lambda_6 \neq 0$ or $\mathbf{Re}\lambda_7 \neq 0$, the condition (**II**) and (**III**);

(3) the conditions (**IV′**) and and (**III**);

(4) the conditions (**IV**).

3.2 Sufficient and necessary conditions

In this subsection, $V_4(\Phi_1, \Phi_2)$ is rewritten as follows $(\lambda_1 > 0, \lambda_2 > 0)$,

$$\begin{aligned}
V_4(\Phi_1, \Phi_2) &= A\rho^2 + B\rho + C = f(\rho), \\
A &= a\phi_1^2\phi_2^2,\ B = b\phi_1\phi_2,\ C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2, \\
a &= \lambda_4 - \mathbf{Re}\lambda_5 + 2(\mathbf{Re}\lambda_5\cos\theta - \mathbf{Im}\lambda_5\sin\theta)\cos\theta \\
&= \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta), \\
b &= 2(\mathbf{Re}\lambda_6\phi_1^2 + \mathbf{Re}\lambda_7\phi_2^2)\cos\theta - 2(\mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2)\sin\theta \\
&= 2|\lambda_6|\phi_1^2\cos(\varphi_6 + \theta) + 2|\lambda_7|\phi_2^2\cos(\varphi_7 + \theta).
\end{aligned} \tag{16}$$

The quadratic function $f(\rho)$ is non-negative about a variable $\rho \in [0, 1]$ if and only if its minimum is non-negative in theinterval $[0, 1]$, and so, its function value is non-negative at the boundary points $\rho = 0, 1$ and the unique extremum point $\rho_0 = -\frac{B}{2A} \in [0, 1](A > 0)$. That is,

$$f(\rho) \geq 0 \Leftrightarrow \begin{cases} f(-\frac{B}{2A}) = \dfrac{4AC - B^2}{4A} \geq 0, -\frac{B}{2A} \in [0, 1], \\ f(0) \geq 0, \\ f(1) \geq 0. \end{cases}$$

The graph-like of $f(\rho)$ is as shown below:

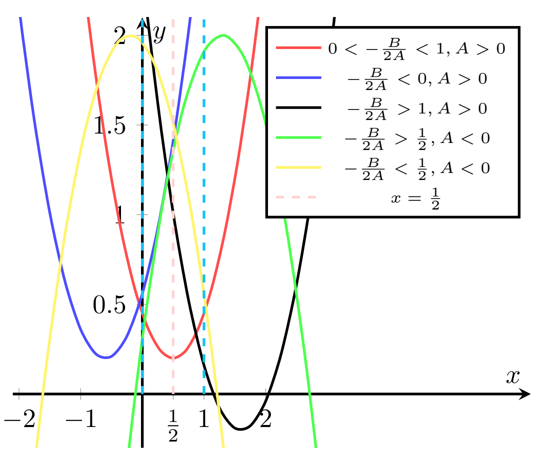

Figure 1: Graph of $f(\rho)$

Proposition 1. $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if

$$\begin{cases} 4AC - B^2 \geq 0, \quad -2A \leq B \leq 0; \\ C \geq 0; \\ A + B + C \geq 0. \end{cases}$$

It is obvious that if $\lambda_6 = \lambda_7 = 0$, then $B = 0$, the symmetry axis $-\frac{B}{2A} = 0$, and hence, $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if

$$C \geq 0 \text{ and } A + C \geq 0.$$

For the 2HDM with $\mathbb{Z}_2$ symmetry [11], the quartic part of the general 2HDM scalar potential is

$$\begin{aligned} V_4^{\mathbb{Z}_2}(\Phi_1, \Phi_2) =& \lambda_1(\Phi_1^*\Phi_1)^2 + \lambda_2(\Phi_2^*\Phi_2)^2 \\ &+ \lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2) + \lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1) \\ &+ \frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2 + \frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2. \end{aligned} \tag{17}$$

Therefore, $V_4^{\mathbb{Z}_2}(\Phi_1, \Phi_2) \geq 0$ if and only if

$$\begin{cases} C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 \geq 0 \\ A + C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta))\phi_1^2\phi_2^2 \geq 0. \end{cases}$$

It is clear that $C \geq 0 \Leftrightarrow \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0$,

$$\begin{aligned} A + C \geq 0 \Leftrightarrow & \lambda_3 + \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta) + 2\sqrt{\lambda_1\lambda_2} \geq 0, \forall\theta \in [0, 2\pi] \\ \Leftrightarrow & \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0. \end{aligned}$$

Corollary 2. $V_4^{\mathbb{Z}_2}(\Phi_1, \Phi_2) \geq 0$ if and only if $\lambda_1 \geq 0, \lambda_2 \geq 0$,

(**V**) $\quad \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0, \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.$

This condition (**V**) is well-known for the inert doublet model [4, 9, 10, 16, 23].

3.3 Necessary conditions

In this subsection, $V_4(\Phi_1, \Phi_2)$ is rewritten as follows ($\lambda_1 > 0, \lambda_2 > 0$),

$$\begin{aligned} &V_4(\Phi_1,\Phi_2) = f(\rho) = A\rho^2 + B\rho + C, \\ &f(1) = A + B + C \\ &\quad = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\mathbf{Re}\lambda_5\cos^2\theta)\phi_1^2\phi_2^2 \\ &\quad\quad + 2(\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1 \\ &\quad\quad - 2(\sin\theta)[(\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 \\ &\quad\quad\quad + \mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2]\phi_1\phi_2. \end{aligned} \tag{18}$$

Obviously, we have

$$f(\rho) \geq 0, \text{ for all } \rho \in [0,1] \Rightarrow f(0) \geq 0, f(1) \geq 0.$$

Then $V_4(\Phi_1, \Phi_2) \geq 0$ implies that $f(0) = C \geq 0$, which is equvalent to

$$(\mathbf{VI}) \qquad \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

This is a necessary condition of the vacuum stability of the general 2HDM potential. Clearly, the other necessary condition is some conditions such that $f(1) = A + B + C \geq 0$. By Eq. (18), it is known that $A + B + C$ may be regarded as a quartic form with two parameters $t = \sin\theta$ and $s = \cos\theta$ with $s^2 + t^2 = 1$. So, when $s = \sin\theta = 0$ and $t = \cos\theta = \pm 1$, the inequality

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathbf{Re}\lambda_6\phi_1^3\phi_2 \pm 2\mathbf{Re}\lambda_7\phi_2^3\phi_1 \geq 0$$

is a necessary condition. From Eq.(7), it follows that the above inequality hold if and only if

(1) $\Delta \leq 0$, $\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1} > 0$;

(2) $\mathbf{Re}\lambda_6 \geq 0$, $\mathbf{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0$;

(3) $\Delta \geq 0$,
$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$,

 (i) $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2}$,

 (ii) $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$,

$\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1} \geq -2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$.

and

(1′) $\Delta \leq 0$, $-\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1} > 0$;

(2′) $-\mathbf{Re}\lambda_6 \geq 0$, $-\mathbf{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0$;

(3′) $\Delta \geq 0$,
$|-\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$,

 (i') $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2}$,

 (ii') $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$,

$-\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1} \geq -2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$.

Which is equivalent to

$$
\begin{aligned}
&\qquad \mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0, \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0;\\
&\qquad \mathbf{Re}\lambda_6 \neq 0 \text{ or } \mathbf{Re}\lambda_7 \neq 0, \Delta \geq 0,\\
&\qquad |\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3+\lambda_4+\mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}\\
&(\mathbf{VII}) \quad (a) - 2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},\\
&\qquad\quad (b)\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2},\\
&\qquad |\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3+\lambda_4+\mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}.
\end{aligned}
$$

If $t = \cos\theta = 0$ and $s = \sin\theta = \pm 1$, the inequality

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathbf{Im}\lambda_6\phi_1^3\phi_2 \pm 2\mathbf{Im}\lambda_7\phi_2^3\phi_1 \geq 0$$

is a necessary condition also, and then, which is equivalent to

$$
\begin{aligned}
&\qquad \mathbf{Im}\lambda_6 = \mathbf{Im}\lambda_7 = 0, \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0;\\
&\qquad \mathbf{Im}\lambda_6 \neq 0 \text{ or } \mathbf{Im}\lambda_7 \neq 0, \Delta' \geq 0,\\
&\qquad |\mathbf{Im}\lambda_6\sqrt{\lambda_2} - \mathbf{Im}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}\\
&(\mathbf{VIII}) \quad (a') - 2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},\\
&\qquad\quad (b')\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2},\\
&\qquad |\mathbf{Im}\lambda_6\sqrt{\lambda_2} + \mathbf{Im}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},
\end{aligned}
$$

where $\Delta' = 4(12\lambda_1\lambda_2 - 12\mathbf{Im}\lambda_6\mathbf{Im}\lambda_7 + (\lambda_3+\lambda_4-\mathbf{Re}\lambda_5)^2)^3 - (72\lambda_1\lambda_2(\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5) + 36\mathbf{Im}\lambda_6\mathbf{Im}\lambda_7(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5) - 2(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5)^3 - 108\lambda_1(\mathbf{Im}\lambda_7)^2 - 108(\mathbf{Im}\lambda_6)^2\lambda_2)^2$.

Applying the Corollary 3.1 of Song and Qi [29] to $V_4(\Phi_1, \Phi_2)$,

$$
\begin{aligned}
V_4(\Phi_1,\Phi_2) =& \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + [\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\varphi_5 + 2\theta)]\phi_1^2\phi_2^2\\
&+ 2|\lambda_6|\phi_1^3\phi_2\rho\cos(\varphi_6+\theta) + 2|\lambda_7|\phi_1\phi_2^3\rho\cos(\varphi_7+\theta),
\end{aligned}
$$

we obtain that $V_4(\Phi_1, \Phi_2) \geq 0$ implies that for all $\rho \in [0, 1]$ and all

$\theta \in [0, 2\pi]$,

$$
\begin{aligned}
0 \leq & 2\left(\frac{1}{4} \times 2|\lambda_6|\rho\cos(\varphi_6+\theta)\right)\sqrt{\lambda_2} + 2\left(\frac{1}{4} \times 2|\lambda_7|\rho\cos(\varphi_7+\theta)\right)\sqrt{\lambda_1} \\
& + \left(3 \times \frac{1}{6}(\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\varphi_5+2\theta)) + \sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2} \\
= & |\lambda_6|\sqrt{\lambda_2}\rho\cos(\varphi_6+\theta) + |\lambda_7|\sqrt{\lambda_1}\rho\cos(\varphi_7+\theta) \\
& + \frac{1}{2}\left(\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\varphi_5+2\theta) + 2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2} \\
= & \mathbf{Re}\lambda_6\sqrt{\lambda_2}\rho\cos\theta + \mathbf{Re}\lambda_7\sqrt{\lambda_2}\rho\cos\theta \\
& + \frac{1}{2}\left(\lambda_3 + \lambda_4\rho^2 + \mathbf{Re}\lambda_5\rho^2\cos 2\theta + 2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2} \\
& - \mathbf{Im}\lambda_6\sqrt{\lambda_2}\rho\sin\theta - \mathbf{Im}\lambda_7\sqrt{\lambda_1}\rho\sin\theta - \frac{1}{2}\mathbf{Im}\lambda_5\sqrt[4]{\lambda_1\lambda_2}\rho^2\sin 2\theta,
\end{aligned}
$$

and then, $\rho = 1$ and $\theta = 0$ or π or $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, the above inequality must holds also. That is,

$$
(\mathbf{IX}) \quad
\begin{aligned}
& (\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2}\big)\sqrt[4]{\lambda_1\lambda_2} \\
& \qquad \pm 2\left(\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}\right) \geq 0 \\
& (\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2}\big)\sqrt[4]{\lambda_1\lambda_2} \\
& \qquad \mp 2\left(\mathbf{Im}\lambda_6\sqrt{\lambda_2} + \mathbf{Im}\lambda_7\sqrt{\lambda_1}\right) \geq 0.
\end{aligned}
$$

Clearly, the condition (**VI**) is obtained if $\rho = 0$.

In summary, the conditions (**VI**), (**VII**), (**VIII**) and (**IX**) are the necessary conditions of the vacuum stability of the general 2HDM potential.

4 Conclusions

By means of the co-positive conditions of a 4th-order symmetry tensor, several analyticial sufficient conditions and necessary conditions are established for the vacuum stability of the general 2HDM potential, respectively. That is,

$$
\text{Four sufficient conditions: } \begin{cases} (1)\ (\mathbf{I}) \text{ and } (\mathbf{III}); \\ (2)\ \mathbf{Re}\lambda_6 \neq 0 \text{ or } \mathbf{Re}\lambda_7 \neq 0, (\mathbf{II}) \text{ and } (\mathbf{III}); \\ (3)\ (\mathbf{IV'}) \text{ and } (\mathbf{III}); \\ (4)\ (\mathbf{IV}). \end{cases}
$$

Four necessary conditions: (**VI**), (**VII**), (**VIII**) and (**IX**).

A sufficient and necessary condition is qualitatively showed for the vacuum stability of the general 2HDM potential, and then, applying it to derive the analyticial necessary conditions for the vacuum stability of the general 2HDM potential. The vacuum stability condition (**V**) of the $\mathbb{Z}_2$ symmetry 2HDM potential is a special case.

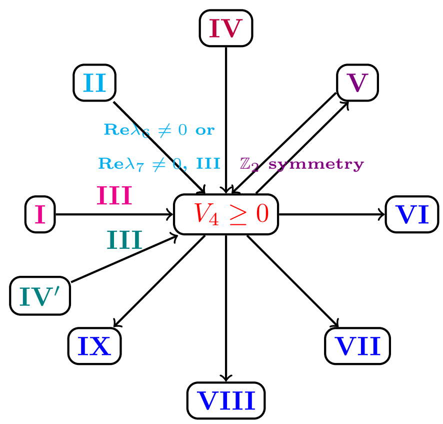

Figure 2: Analyticial conditions and the vacuum stability of the general 2HDM potential

References

[1] T. Lee, A Theory of Spontaneous T Violation, Phys. Rev. D 8, 1226(1973).

[2] T. Lee, CP nonconservation and spontaneous symmetry breaking, Physics Reports, 9(2) 143-177(1974).

[3] S. Weinberg, Gauge Theory of CP Nonconservation, Phys. Rev. Lett. 37, 657 (1976).

[4] N. G. Deshpande, E. Ma, Pattern of symmetry breaking with two Higgs doublets, Phys. Rev. D 18, 2574-2576(1978).

[5] Y.L. Wu and L. Wolfenstein, Sources of CP Violation in the Two-Higgs-Doublet Model, Phys. Rev. Lett. 73, 1762(1994).

[6] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher, and J.P. Silva, Theory and phenomenology of two-Higgs-doublet models, Phys. Rep. 516, 1-102 (2012).

[7] I. P. Ivanov, J.P. Silva, Tree-level metastability bounds for the most general two Higgs doublet model. Phys. Rev. D 92, 055017(2015).

[8] F.J. Botella, J.P. Silva, Jarlskog-like invariants for theories with scalars and fermions, Phys. Rev. D 51 3870(1995).

[9] I.F. Ginzburg, M. Krawczyk, Symmetries of two Higgs doublet model and CP violation, Phys. Rev. D 72, 115013(2005).

[10] K. Klimenko, Conditions for certain Higgs potentials to be bounded below, Theor. Math. Phys. 62, 58-65 (1985).

[11] M. Nebot, Bounded masses in two Higgs doublets models, spontaneous CP violation and $\mathbb{Z}_2$ symmetry, Phys. Rev. D 102, 115002 (2020).

[12] S. Nie, M. Sher, Vacuum stability bounds in the two-Higgs doublet model, Phys. Lett. B 449(1-2), 89-92 (1999).

[13] S. Kanemura, T. Kasai, Y. Okada, Mass bounds of the lightest CP-even Higgs boson in the two-Higgs-doublet model, Phys. Lett. B 471(2-3), 182-190 (1999).

[14] D. Eriksson, J. Rathsman, O. Stal, 2HDMC-two-Higgs-doublet model calculator, Comput. Phys. Commun. 181(1), 189-205 (2010); Erratum, Comput. Phys. Commun. .181(5), (2010).

[15] K. Kannike, Vacuum stability of a general scalar potential of a few fields. Eur. Phys. J. C, 76, 324(2016); Erratum, Eur. Phys. J. C, 78, 355(2018).

[16] K. Kannike, Vacuum stability conditions from copositivity criteria. Eur. Phys. J. C, 72, 2093 (2012).

[17] K. Kannike, Vacuum stability conditions and potential minima for a matrix representation in lightcone orbit space. Eur. Phys. J. C 81, 940 (2021).

[18] G. Chauhan, Vacuum stability and symmetry breaking in left-right symmetric model. J. High Energ. Phys. 2019, 137 (2019).

[19] Y. Song, Co-positivity of tensors and Stability conditions of two Higgs potentials, arXiv:2203.11462, Submitted on 22 Mar 2022.

[20] A. Barroso, P.M. Ferreira, I.P. Ivanov, and R. Santos, Metastability bounds on the two Higgs doublet model, J. High Energy Phys. 06, 045(2013).

[21] I.P. Ivanov, Minkowski space structure of the Higgs potential in the two-Higgs-doublet model, Phys. Rev. D 75, 035001; Erratum Phys. Rev. D 76, 039902 (2007).

[22] H. Bahl, M. Carena, N. M. Coyle, A. Ireland, Carlos E.M. Wagner, New Tools for Dissecting the General 2HDM, arXiv:2210.00024 (2022)

[23] M. Maniatisa, A. von Manteuffelb, O. Nachtmannc, F. Nagel, Stability and symmetry breaking in the general two-Higgs-doublet model, Eur. Phys. J. C 48, 805-823 (2006).

[24] L.E. Andersson, G. Chang, and T. Elfving, Criteria for copositive matrices using simplices and barycentric coordinates, Linear Algebra Appl. 5, 9-30(1995)

[25] K.P. Hadeler, On copositive matrices, Linear Algebra Appl. 49, 79-89(1983).

[26] E. Nadler, Nonnegativity of bivariate quadratic functions on a triangle, Comput. Aided Geom. D. 9, 195-205(1992).

[27] J. Liu, Y. Song, Copositivity for 3rd order symmetric tensors and applications, Bull. Malays. Math. Sci. Soc. 45(1), 133-152(2022).

[28] Y. Song, X. Li, Copositivity for a class of fourth order symmetric tensors given by scalar dark matter, J. Optim Theory Appl. 195, 334?46 (2022)

[29] Y. Song, L. Qi, Analytical expressions of copositivity for fourth-order symmetric tensors, Analy. Appl., 19(5), 779-800(2021)

[30] Y. Song, Positive definiteness for 4th order symmetric tensors and applications, Anal. Math. Phys. 11, 10(2021)

[31] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput., 40(6) 1302-1324(2005)

[32] L. Qi, Symmetric Nonnegative Tensors and Copositive Tensors, Linear Algebra Appl., 439,228-238(2013)

[33] L. Qi, Y. Song, X. Zhang, Positivity Conditions for Cubic, Quartic and Quintic Polynomials, J. Nonlinear Convex Anal. 23(2), 191-213(2022)

[34] Y. Song, L. Qi, Necessary and sufficient conditions for copositive tensors, Linear Multilinear A 63(1), 120-131(2015).

[35] L. Qi, H. Chen, Y. Chen, Tensor Eigenvalues and Their Applications, Springer Singapore, 2018.

[36] L. Qi, Z. Luo, Tensor Analysis: Spectral Theory and Special Tensors, SIAM, Philadelpia 2017.

[37] G. Ulrich, L.T. Watson, Positivity conditions for quartic polynomials. SIAM J. Sci. Comput. 15, 528-544(1994)